Invariants for Correlations of Velocity Differences in Turbulent Fields

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The phenomenology of the scaling behavior of higher order structure functions of velocity differences across a scale R in turbulence should be built around the irreducible representations of the rotation symmetry group. Every irreducible representation is associated with a scalar function of R which may exhibit different scaling exponents. The common practice of using moments of longitudinal and transverse fluctuations mixes different scalar functions and therefore may mix different scaling exponents. It is shown explicitly how to extract pure scaling exponents for correlations functions of arbitrary orders.

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Traditional measurements of anomalous scaling in turbulence are based on hot wire technology which yields information about the longitudinal components of the velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ [1]. Accordingly it is customary to consider the structure functions of longitudinal velocity differences:

$$S_n(R) = \langle [\delta u_l(\boldsymbol{r},\boldsymbol{R},t)]^n \rangle \ , \tag{1}$$

$$\delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{R},t) \equiv \boldsymbol{u}(\boldsymbol{r}+\boldsymbol{R},t) - \boldsymbol{u}(\boldsymbol{r},t) \ , \tag{2}$$

$$\delta u_\ell(\boldsymbol{r},\boldsymbol{R},t) \equiv \delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{R},t)\cdot\boldsymbol{R}/R \ . \tag{3}$$

It is well known that these structure functions appear to scale with scaling exponent ζ_n^ℓ which are anomalous (nonlinear functions of n):

$$S_n(R) \sim R^{\zeta_n^\ell} \ . \tag{4}$$

Only recently has it become feasible, due to advances in experimental technology [2–4], and even more so in computational methods [5–7], to measure other components of the velocity field. In particular a number of groups have focused on the transverse components

$$\delta\boldsymbol{u}_t(\boldsymbol{r},\boldsymbol{R},t) \equiv \delta\boldsymbol{u}(\boldsymbol{r},\boldsymbol{R},t) - \delta u_\ell(\boldsymbol{r},\boldsymbol{R},t)\boldsymbol{R}/R. \tag{5}$$

These groups studied the scaling exponents of the transverse structure functions

$$T_n(R) = \langle |\delta\boldsymbol{u}_t(\boldsymbol{r},\boldsymbol{R},t)|^n \rangle \sim R^{\zeta_n^t} \ . \tag{6}$$

Two sets of measurements appear to imply that the scaling exponents ζ_n^ℓ are the same as ζ_n^t within experimental uncertainty [2,3], whereas other numerical [5–7] and experimental [4] studies indicate the opposite, i.e. that ζ_n^t are significantly smaller than ζ_n^ℓ for $n \geq 4$.

The main point of this Letter is to demonstrate that higher order structure functions of longitudinal and transverse moments are not likely to exhibit clean scaling behavior, since they mix different scalar functions of R which may scale with different scaling exponents. In experimental and numerical studies in which all the components of the velocity field are available it is advisable to consider moments that are invariant under rotations [8]; such invariants are expected to scale with pure scaling exponents that can be extracted from the data.

The problem of mixing of different scalar functions does not exist for the second and third order moments of the longitudinal and transverse components. It is worthwhile to go in detail through the analysis of the 2nd order moment in order to see why the longitudinal and transverse components are not a good choice, and why at the end it does not matter at this order. In an isotropic homogeneous medium without helicity (with inversion symmetry) the relevant symmetry group is the rotation group SO(3) whose irreducible representations can be expressed using the spherical harmonics $Y_{\ell,m}$. The most general form of the second order moment of velocity differences has contributions from $\ell = 0$ and 2:

$$\langle \delta u^\alpha(\boldsymbol{r},\boldsymbol{R},t)\delta u^\beta(\boldsymbol{r},\boldsymbol{R},t)\rangle = \delta_{\alpha\beta}a_0(R) + \left[\delta_{\alpha\beta} - \frac{3R_\alpha R_\beta}{R^2}\right] a_2(R) \ . \tag{7}$$

The coefficients in this expression carry the index ℓ, multiplying terms that are irreducible representations of the rotation group of dimension $2\ell+1$. The dimension of the irreducible representation is the number of tensor components that transform to one another upon rotation of the system of coordinates. All the tensor components of a given irreducible representation with a given value of ℓ must have the same coefficient which depends only on R. On the other hand the scalar functions $a_0(R)$ and $a_2(R)$ may have different scaling exponents.

Computing now the longitudinal and transverse moments we find

$$\langle \delta u_\ell \delta u_\ell \rangle = \frac{R^\alpha R^\beta}{R^2}\langle \delta u^\alpha \delta u^\beta \rangle = a_0(R) - 2a_2(R), \tag{8}$$

$$\langle \delta\boldsymbol{u}_t\cdot\delta\boldsymbol{u}_t \rangle = \langle |\delta\boldsymbol{u}|^2 \rangle - \langle \delta u_\ell \delta u_\ell \rangle = 2a_0(R) + 2a_2(R). \tag{9}$$

Obviously these moments mix the two scalar functions with different weights. Fortunately the incompressibility constraint forces $a_0(R)$ and $a_2(R)$ to have the same scaling exponent. We compute

$$\frac{\partial}{\partial R^\alpha}\langle \delta u^\alpha \delta u^\beta\rangle = \frac{R^\beta}{R}[\frac{da_0}{dR} - 2\frac{da_2}{dR} - 6\frac{a_2(R)}{R}] = 0 \ , \quad (10)$$

meaning that the two functions must have the same R scaling, and therefore also the 2nd order longitudinal and transverse components scale with the same exponents.

The purity (and identity) of exponents of longitudinal and transverse fluctuations also holds for the third order moments. The most general form of the third order tensor $\langle \delta u^\alpha \delta u^\beta \delta u^\gamma\rangle$ has contributions from $\ell = 1$ and 3:

$$\langle \delta u^\alpha \delta u^\beta \delta u^\gamma\rangle = b_1(R)[\delta_{\alpha\beta}R^\gamma + \delta_{\alpha\gamma}R^\beta + \delta_{\beta\gamma}R^\alpha] \\ +b_3(R)[\delta_{\alpha\beta}R^\gamma + \delta_{\alpha\gamma}R^\beta + \delta_{\beta\gamma}R^\alpha - 5R^\alpha R^\beta R^\gamma/R^2] \ .$$

We again have two distinct scalar functions, each multiplying a rotationally invariant form, and scaling with potentially different scaling exponents. Nevertheless, the incompressibility constraint provides one relation among the scalar functions, leaving us with one unknown. Kolmogorov showed [9] that the rate of energy dissipation fixes the value of the remaining unknown. The form of the 3rd order tensor is thus fully determined, and a calculation shows that $\left\langle |\delta u_\ell(\boldsymbol{r},\boldsymbol{R},t)|^3\right\rangle \sim \left\langle |\delta u_t(\boldsymbol{r},\boldsymbol{R},t)|^3\right\rangle \sim R$.

The first nontrivial example is the 4th order tensor $\langle \delta u^\alpha \delta u^\beta \delta u^\gamma \delta u^\delta\rangle$. The most general form of this tensor has contributions with $\ell = 0, 2$ and 4:

$$\langle \delta u^\alpha \delta u^\beta \delta u^\gamma \delta u^\delta\rangle = c_0(R)D_0^{\alpha\beta\gamma\delta} + c_2(R)D_2^{\alpha\beta\gamma\delta} \\ + c_4(R)D_4^{\alpha\beta\gamma\delta} \ , \quad (11)$$

where

$$D_0^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{45}}\left[\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right] \ , \quad (12)$$

$$D_2^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{28}R^2}\Big[R^\alpha R^\beta\delta_{\gamma\delta} + R^\alpha R^\gamma\delta_{\beta\delta} + R^\alpha R^\delta\delta_{\beta\gamma} \\ +R^\beta R^\gamma\delta_{\alpha\delta} + R^\beta R^\delta\delta_{\alpha\gamma} + R^\gamma R^\delta\delta_{\alpha\beta}\Big] - \sqrt{\frac{5}{7}}D_0^{\alpha\beta\gamma\delta} \ , \quad (13)$$

$$D_4^{\alpha\beta\gamma\delta} = \sqrt{\frac{35}{8}}\frac{R^\alpha R^\beta R^\gamma R^\delta}{R^4} - \sqrt{\frac{5}{2}}D_2^{\alpha\beta\gamma\delta} - \sqrt{\frac{7}{8}}D_0^{\alpha\beta\gamma\delta} \ . \quad (14)$$

We see that in this case we have three independent scalar functions of R, i.e. $c_0(R), c_2(R)$ and $c_4(R)$, which in principle may have different scaling exponents. In this case the incompressibility constraint furnishes no relation between these functions; the reason is that there exist contributions in this tensor like $\langle u^\alpha(\boldsymbol{r},t)u^\beta(\boldsymbol{r},t)u^\gamma(\boldsymbol{r}+\boldsymbol{R},t)u^\delta(\boldsymbol{r}+\boldsymbol{R},t)\rangle$, and the divergence of such a contribution (with summation on any tensor index) is not zero. In fact, incompressibility no longer places constraints for any of the higher order correlation functions for similar reasons. We note that there is no known way to justify why the three scalar functions should have the same dependence on R. We can compute now the longitudinal and transverse 4th order moments:

$$\langle(\delta u_\ell)^4\rangle \equiv \frac{1}{R^4}R^\alpha R^\beta R^\gamma R^\delta\langle \delta u^\alpha \delta u^\beta \delta u^\gamma \delta u^\delta\rangle \quad (15)$$

$$\langle|\delta \boldsymbol{u}_t|^4\rangle \equiv \langle(\delta\boldsymbol{u}_t\cdot\delta\boldsymbol{u}_t)^2\rangle \quad (16)$$
$$= (\delta_{\alpha\beta} - \frac{R^\alpha R^\beta}{R^2})(\delta_{\gamma\delta} - \frac{R^\gamma R^\delta}{R^2})\langle \delta u^\alpha \delta u^\beta \delta u^\gamma \delta u^\delta\rangle \ .$$

A calculation yields

$$\langle(\delta u_\ell)^4\rangle = \frac{c_0(R)}{\sqrt5} + \frac{2c_2(R)}{\sqrt7} + c_4(R)\sqrt{\frac{8}{35}} \ , \quad (17)$$

$$\langle|\delta \boldsymbol{u}_t|^4\rangle = \frac{8c_0(R)}{3\sqrt5} - \frac{8c_2(R)}{3\sqrt7} + c_4(R)\sqrt{\frac{8}{35}} \ . \quad (18)$$

We see that these components mix the three scalar functions with different coefficients. There are two possibilities: either all the scalar functions have the same leading scaling exponent, or they have different scaling exponents. In the first case it is obvious that the longitudinal and transverse moments share the same scaling exponents. In the second case, for a sufficiently long inertial range, and for $R \ll L$ where L is the outer scale of turbulence, the smallest exponent will dominate the scaling of both moments. Asymptotically the two moments are expected to have the same scaling behavior. However, if the three functions have different (leading) exponents, data with limited scaling range may lead to the erroneous conclusion that these moments have different scaling exponents. It should be stressed that the amplitudes of the three scalar functions *may be not universal*, and different experiments may lead to different weights in this mixed representation. This may lead to a possible confusion or to conflicting results as seen in refs. [2]- [7].

The more rational procedure that presents itself in light of this discussion is to compute the scaling behavior of the *invariant* scalar functions which are associated with the higher order tensors. To achieve this we use the orthonormality of the irreducible representations, and observe that

$$c_0(R) = D_0^{\alpha\beta\gamma\delta}\langle \delta u^\alpha \delta u^\beta \delta u^\gamma \delta u^\delta\rangle \ , \quad (19)$$
$$c_2(R) = D_2^{\alpha\beta\gamma\delta}\langle \delta u^\alpha \delta u^\beta \delta u^\gamma \delta u^\delta\rangle \ , \quad (20)$$
$$c_4(R) = D_4^{\alpha\beta\gamma\delta}\langle \delta u^\alpha \delta u^\beta \delta u^\gamma \delta u^\delta\rangle \ . \quad (21)$$

Using the explicit form of the irreducible representations (12)-(13) we can evaluate these functions and find

$$c_0(R) \propto \langle|\delta\boldsymbol{u}|^4 P_0\left(\frac{\delta u_\ell}{|\delta\boldsymbol{u}|}\right)\rangle \propto \langle|\delta\boldsymbol{u}|^4\rangle \ , \quad (22)$$

$$c_2(R) \propto \langle|\delta\boldsymbol{u}|^4 P_2\left(\frac{\delta u_\ell}{|\delta\boldsymbol{u}|}\right)\rangle \\ \propto \langle|\delta\boldsymbol{u}|^2\left[3(\delta u_\ell)^2 - |\delta\boldsymbol{u}|^2\right]\rangle \ , \quad (23)$$

$$c_4(R) \propto \langle|\delta\boldsymbol{u}|^4 P_4\left(\frac{\delta u_\ell}{|\delta\boldsymbol{u}|}\right)\rangle \\ \propto \langle 35\delta u_\ell^4 - 30\delta u_\ell^2|\delta\boldsymbol{u}|^2 + 3|\delta\boldsymbol{u}|^4\rangle \ , \quad (24)$$

where P_ℓ are the standard Legendre polynomials of order ℓ. We see that our scalar functions can be represented as particular combinations of transverse and longitudinal fluctuations. With data from a turbulent field $\boldsymbol{u}(\boldsymbol{r},t)$ one can compute in this way each of the independent scalar functions. Plotting them in double logarithmic plots (to get rid of the nonuniversal amplitudes) one has a good chance of extracting pure scaling behavior. After doing so one can return to the analysis of the longitudinal and transverse components with some understanding of the leading and subleading scaling exponents, to control the apparent scaling behavior in limited scaling ranges.

These considerations are readily extended to higher order moments. The nth order tensor of velocity differences across a scalar R will have $n/2+1$ invariant scalar functions for n odd, and $(n+1)/2$ invariant functions for n even. There is no need to write down the explicit form of the irreducible representations, since the structure exhibited by Eqs.(22) -(24) repeats at all orders. In other words, the independent scalar function $d^n_\ell(R)$ which is the function associated with the irreducible representation of order ℓ in the nth rank tensor of velocity differences can be written in general as

$$d^n_\ell(R) \propto \left\langle |\delta\boldsymbol{u}|^n P_\ell\left(\frac{\delta u_\ell}{|\delta\boldsymbol{u}|}\right)\right\rangle \ , \quad \ell \le n \ , \tag{25}$$

where ℓ has the same parity as n. Thus by simply examining the Legendre polynomials in any textbook one can determine the precise combination of longitudinal and transverse fluctuations that is expected to scale with a pure exponents for any order n.

In conclusion, it appears extremely worthwhile, in light of the growing abundance of high quality data on full turbulent velocity fields, to implement the approach detailed above. Since one confronts limited scaling ranges in most applications, it is mandatory to attempt to separate leading from subleading scaling contributions in order to be able to make substantial conclusions about the numerical values of scaling exponents. The procedure outlined above goes some way in this direction.

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